BLOW-UP PHENOMENA FOR THE YAMABE EQUATION

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Abstract. Let (M, g) be compact Riemannian manifold of dimension $n \geq 3$. A well-known conjecture states that the set of constant scalar curvature metrics in the conformal class of g is compact unless (M, g) is conformally equivalent to the round sphere. In this paper, we construct counterexamples to this conjecture in dimensions $n \geq 52$.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem is concerned with finding metrics of constant scalar curvature in the conformal class of g. This problem can be reduced to a semi-linear elliptic PDE. Indeed, the metric $u^{\frac{4}{n-2}}\, g$ has constant scalar curvature c if and only if

$$\frac{4(n-1)}{n-2}\,\Delta_g u - R_g\, u + c\, u^{\frac{n+2}{n-2}} = 0, \tag{1}$$

where Δ_g is the Laplace operator with respect to g and R_g denotes the scalar curvature of g. Clearly, every solution of (1) is a critical point of the functional

$$E_g(u) = \frac{\int_M \big(\frac{4(n-1)}{n-2}\,|du|_g^2 + R_g\, u^2\big)\, dvol_g}{\big(\int_M u^{\frac{2n}{n-2}}\, dvol_g\big)^{\frac{n-2}{n}}}. \tag{2}$$

It is well-known that the PDE (1) has at least one positive solution for any choice of (M, g). If $n \geq 6$ and (M, g) is not locally conformally flat, this follows from results of T. Aubin [3]. The remaining cases were solved by R. Schoen using the positive mass theorem [16].

Solutions to (1) are not usually unique. As an example, consider the product metric on $S^1(L) \times S^{n-1}(1)$. If L is sufficiently small, then the Yamabe PDE has a unique solution. On the other hand, there are many non-minimizing solutions if L is large. D. Pollack [14] has used gluing techniques to construct high energy solutions on more general background manifolds: given any conformal class with positive Yamabe constant and any positive integer N, there exists a new conformal class which is close to the original

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one in the C^0-norm, and contains at least N metrics of constant scalar curvature (see [14], Theorem 0.1).

It is an interesting question whether the set of all solutions to the Yamabe PDE is compact (in the C^2-topology, say). A well-known conjecture states that this should be true unless (M, g) is conformally equivalent to the round sphere (see [17],[18],[19]). This conjecture has been verified in low dimensions and in the locally conformally flat case: if (M, g) is locally conformally flat, compactness follows from work of R. Schoen [17],[18]. Moreover, Schoen proposed a strategy for proving the conjecture in the non-locally conformally flat case based on the Pohozaev identity. In [12], Y.Y. Li and M. Zhu [12] followed this strategy to prove compactness in dimension 3. O. Druet [7] proved the conjecture in dimensions 4 and 5. Recently, F. Marques [13] showed that compactness holds up to dimension 7. The same result was obtained independently by Y.Y. Li and L. Zhang [11]. Moreover, Li and Zhang showed that compactness holds in all dimensions provided that $|W_g(p)| + |\nabla W_g(p)| > 0$ for all $p \in M$. M. Khuri, F. Marques, and R. Schoen [10] proved compactness up to dimension 24, assuming that the positive mass theorem holds.[1]

In this paper, we address the opposite question: is it possible to construct Riemannian manifolds (M, g) such that the set of constant scalar curvature metrics in the conformal class of g is non-compact? So far, the only known examples where compactness fails involve non-smooth background metrics. The first result in this direction was established by A. Ambrosetti and A. Malchiodi [2]. This result was subsequently improved by M. Berti and A. Malchiodi [6]. Given positive integers n and k such that $k \geq 2$ and $n \geq 4k + 3$, Berti and Malchiodi showed that there exists a Riemannian metric g on S^n (of class C^k) for which the set of solutions to the Yamabe PDE (1) fails to be compact (see [6], Theorem 1.1). A survey of these results can be found in [1]. Recently, O. Druet and E. Hebey [8] showed that blow-up can occur for problems of the form $Lu + c\, u^{\frac{n+2}{n-2}} = 0$, where L is a lower order perturbation of the conformal Laplacian on S^n.

We improve the results of Berti and Malchiodi by showing that the set of solutions to the Yamabe PDE (1) can fail to be compact even if the background metric g is C^∞ smooth. In the examples we construct, the blowing-up sequence develops a singularity consisting of exactly one bubble.

Theorem. *Assume that $n \geq 52$. Then there exists a Riemannian metric g on S^n (of class C^∞) and a sequence of positive functions $v_\nu \in C^\infty(S^n)$ ($\nu \in \mathbb{N}$) with the following properties:*

(i) *g is not conformally flat*
(ii) *v_ν is a solution of the Yamabe PDE (1) for all $\nu \in \mathbb{N}$*
(iii) *$E_g(v_\nu) < Y(S^n)$ for all $\nu \in \mathbb{N}$, and $E_g(v_\nu) \to Y(S^n)$ as $\nu \to \infty$*
(iv) *$\sup_{S^n} v_\nu \to \infty$ as $\nu \to \infty$*

[1] T. Aubin has recently claimed a general compactness theorem in all dimensions [4],[5]. We have, however, been unable to verify some of the arguments in [4].

(Here, $Y(S^n)$ denotes the Yamabe energy of the round metric on S^n.)

Let us sketch the main steps involved in the proof of Theorem 1. For convenience, we will work on $\mathbb{R}^n$ instead of S^n. Let g be a smooth metric on $\mathbb{R}^n$ which agrees with the Euclidean metric outside a ball of radius 1. We will assume throughout the paper that $\det g(x) = 1$ for all $x \in \mathbb{R}^n$, so that the volume form associated with g agrees with the Euclidean volume form.

Our goal is to construct solutions to the Yamabe PDE on $(\mathbb{R}^n, g)$. In Section 2, we show that this problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \varepsilon)$, where ξ is a vector in $\mathbb{R}^n$ and ε is a positive real number. This idea has been used by many authors (see, e.g., [2] or [6]). In Section 3, we show that the function $\mathcal{F}_g(\xi, \varepsilon)$ can be approximated by an auxiliary function $F(\xi, \varepsilon)$. In Section 4, we prove that the function $F(\xi, \varepsilon)$ has a critical point, which is a strict local minimum. Finally, in Section 5, we use a perturbation argument to construct critical points of the function $\mathcal{F}_g(\xi, \varepsilon)$. From this the main result follows.

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2. Lyapunov-Schmidt reduction

Let

$$\mathcal{E} = \Big\{ w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |dw|^2 < \infty \Big\}.$$

By Sobolev's inequality, there exists a constant K, depending only on n, such that

$$\Big(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \Big)^{\frac{n-2}{n}} \leq K \int_{\mathbb{R}^n} |dw|^2$$

for all $w \in \mathcal{E}$. We define a norm on $\mathcal{E}$ by $\|w\|_{\mathcal{E}}^2 = \int_{\mathbb{R}^n} |dw|^2$. It is easy to see that $\mathcal{E}$, equipped with this norm, is complete.

Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, we define a function $u_{(\xi,\varepsilon)} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{(\xi,\varepsilon)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \Big)^{\frac{n-2}{2}}.$$

The function $u_{(\xi,\varepsilon)}$ satisfies the elliptic PDE

$$\Delta u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} = 0.$$

It is well known that

$$\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} = \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}}$$

for all $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. We next define

$$\varphi_{(\xi,\varepsilon,0)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2}\Big)^{\frac{n+2}{2}} \frac{\varepsilon^2 - |x - \xi|^2}{\varepsilon^2 + |x - \xi|^2}$$

and

$$\varphi_{(\xi,\varepsilon,k)}(x) = \Big(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2}\Big)^{\frac{n+2}{2}} \frac{2\varepsilon\,(x_k - \xi_k)}{\varepsilon^2 + |x - \xi|^2}$$

for $k = 1, \ldots, n$. It is easy to see that the norm $\|\varphi_{(\xi,\varepsilon,k)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$ is constant in ξ and ε. Finally, we define a closed subspace $\mathcal{E}_{(\xi,\varepsilon)} \subset \mathcal{E}$ by

$$\mathcal{E}_{(\xi,\varepsilon)} = \bigg\{w \in \mathcal{E} : \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, w = 0 \quad \text{for } k = 0, 1, \ldots, n\bigg\}.$$

Clearly, $u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 1. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq 1$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq 1$. There exists a constant C, depending only on n, such that*

$$\Big\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\,\alpha$$

for all pairs $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$.

Proof. Using the pointwise estimate

$$\begin{aligned}&\Big|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big| \\ &\leq C\,|h|\,|\partial^2 u_{(\xi,\varepsilon)}| + C\,|\partial h|\,|\partial u_{(\xi,\varepsilon)}| + C\,(|\partial^2 h| + |\partial h|^2)\, u_{(\xi,\varepsilon)},\end{aligned}$$

we obtain

$$\begin{aligned}&\Big\|\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ &\leq C\,\|h\|_{L^\infty(\mathbb{R}^n)}\,\|\partial^2 u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} + C\,\|\partial h\|_{L^n(\mathbb{R}^n)}\,\|\partial u_{(\xi,\varepsilon)}\|_{L^2(\mathbb{R}^n)} \\ &+ C\,(\|\partial^2 h\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} + \|\partial h\|_{L^n(\mathbb{R}^n)}^2)\,\|u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ &\leq C\,\alpha.\end{aligned}$$

This proves the assertion.

Proposition 2. *There exists a positive constant θ, depending only on n, such that*

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(|dw|^2 - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w^2\Big) \\ &\geq 2\theta\,\|w\|_{\mathcal{E}}^2 - \frac{16n^2}{\theta}\,\bigg(\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\, w\bigg)^2\end{aligned}$$

for all $w \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 2 follows from an analysis of the eigenvalues of the Laplace operator on S^n. The details can be found in [15].

Corollary 3. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. There exists a positive constant $\alpha_0 \leq 1$, depending only on n, with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$, then we have*

$$\Big(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \Big)^{\frac{n-2}{n}} \leq 2K \int_{\mathbb{R}^n} \Big(|dw|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, w^2 \Big)$$

for all $w \in \mathcal{E}$ and

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big(|dw|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, w^2 - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w^2 \Big) \\ &\geq \theta \, \|w\|_{\mathcal{E}}^2 - \frac{1}{\theta} \Big(\int_{\mathbb{R}^n} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \, w \Big)^2 \end{aligned}$$

for all $w \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proof. Using Proposition 1 and Hölder's inequality, we obtain

$$\begin{aligned} &\Big| \int_{\mathbb{R}^n} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \, w \Big| \\ &\geq 4n \, \Big| \int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \, w \Big| - C \, \alpha_0 \, \|w\|_{\mathcal{E}}. \end{aligned}$$

This implies

$$\begin{aligned} &\Big(\int_{\mathbb{R}^n} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \, w \Big)^2 \\ &\geq 16n^2 \, \Big(\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \, w \Big)^2 - \theta^2 \, \|w\|_{\mathcal{E}}^2 \end{aligned}$$

if α_0 is sufficiently small. Hence, the assertion follows from Proposition 2.

Proposition 4. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq 1$. Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$ and any function $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, there exists a unique function $w \in \mathcal{E}_{(\xi,\varepsilon)}$ such that*

$$\int_{\mathbb{R}^n} \Big(\langle dw, d\psi \rangle_g + \frac{n-2}{4(n-1)} \, R_g \, w \, \psi - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w \, \psi \Big) = \int_{\mathbb{R}^n} f \, \psi$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Moreover, we have $\|w\|_{\mathcal{E}} \leq C \, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$, where C is a constant that depends only on n.

Proof. Suppose that $w \in \mathcal{E}_{(\xi,\varepsilon)}$ and

$$\int_{\mathbb{R}^n} \Big(\langle dw, d\psi \rangle_g + \frac{n-2}{4(n-1)} \, R_g \, w \, \psi - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w \, \psi \Big) = \int_{\mathbb{R}^n} f \, \psi$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. This implies

$$\int_{\mathbb{R}^n} \Big(|dw|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, w^2 - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w^2 \Big) = \int_{\mathbb{R}^n} f \, w$$

and

$$\int_{\mathbb{R}^n} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \, w = - \int_{\mathbb{R}^n} u_{(\xi,\varepsilon)} \, f.$$

Using Corollary 3, we obtain

$$\begin{aligned}
\theta \, \|w\|_{\mathcal{E}}^2 &\leq \int_{\mathbb{R}^n} \Big(|dw|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, w^2 - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w^2 \Big) \\
&\quad + \frac{1}{\theta} \bigg(\int_{\mathbb{R}^n} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \, w \bigg)^2 \\
&\leq \bigg(\int_{\mathbb{R}^n} |f|^{\frac{2n}{n+2}} \bigg)^{\frac{n+2}{2n}} \, \bigg(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \bigg)^{\frac{n-2}{2n}} \\
&\quad + \frac{1}{\theta} \bigg(\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} \bigg)^{\frac{n-2}{n}} \, \bigg(\int_{\mathbb{R}^n} |f|^{\frac{2n}{n+2}} \bigg)^{\frac{n+2}{n}} \\
&\leq K^{\frac{1}{2}} \, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \, \|w\|_{\mathcal{E}} + \frac{1}{\theta} \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n-2}{2}} \, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2.
\end{aligned}$$

Hence, it follows from Young's inequality that

$$\frac{\theta}{2} \, \|w\|_{\mathcal{E}}^2 \leq \frac{K}{2\theta} \, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2 + \frac{1}{\theta} \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n-2}{2}} \, \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2.$$

From this the uniqueness statement follows easily.

In order to prove the existence part, it suffices to minimize the functional

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(|dw|_g^2 + \frac{n-2}{4(n-1)} \, R_g \, w^2 - n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w^2 - 2fw \Big) \\
&+ \frac{1}{\theta} \bigg(\int_{\mathbb{R}^n} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \, w \bigg)^2
\end{aligned}$$

over all functions $w \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 5. *Consider a Riemannian metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$. Moreover, let $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. There exists a positive constant $\alpha_1 \leq \alpha_0$, depending only on n, with the following property:*

if $|h(x)|+|\partial h(x)|+|\partial^2 h(x)| \leq \alpha_1$ *for all* $x \in \mathbb{R}^n$*, then there exists a function* $v_{(\xi,\varepsilon)} \in \mathcal{E}$ *such that* $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ *and*

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} \, R_g \, v_{(\xi,\varepsilon)} \, \psi - n(n-2) \, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} \, v_{(\xi,\varepsilon)} \, \psi \Big) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$*. Moreover, we have the estimate*

$$\begin{aligned} &\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{\mathcal{E}} \\ &\leq C \, \Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n-2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \end{aligned}$$

where C *is a constant that depends only on* n*.*

Proof. Let $G_{(\xi,\varepsilon)} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to \mathcal{E}_{(\xi,\varepsilon)}$ be the solution operator constructed in Proposition 4. We define a nonlinear mapping $\Phi_{(\xi,\varepsilon)} : \mathcal{E}_{(\xi,\varepsilon)} \to \mathcal{E}_{(\xi,\varepsilon)}$ by

$$\begin{aligned} &\Phi_{(\xi,\varepsilon)}(w) \\ &= G_{(\xi,\varepsilon)} \Big(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n-2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big) \\ &+ n(n-2) \, G_{(\xi,\varepsilon)} \Big(|u_{(\xi,\varepsilon)} + w|^{\frac{4}{n-2}} \, (u_{(\xi,\varepsilon)} + w) - u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w \Big). \end{aligned}$$

It follows from Proposition 1 that $\|\Phi_{(\xi,\varepsilon)}(0)\|_{\mathcal{E}} \leq C \, \alpha_1$. Using the pointwise estimate

$$\begin{aligned} &\Big| |u_{(\xi,\varepsilon)} + w|^{\frac{4}{n-2}} \, (u_{(\xi,\varepsilon)} + w) - |u_{(\xi,\varepsilon)} + \tilde{w}|^{\frac{4}{n-2}} \, (u_{(\xi,\varepsilon)} + \tilde{w}) \\ &\qquad - \frac{n+2}{n-2} \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, (w - \tilde{w}) \Big| \\ &\leq C \, (|w|^{\frac{4}{n-2}} + |\tilde{w}|^{\frac{4}{n-2}}) \, |w - \tilde{w}|, \end{aligned}$$

we obtain

$$\begin{aligned} &\|\Phi_{(\xi,\varepsilon)}(w) - \Phi_{(\xi,\varepsilon)}(\tilde{w})\|_{\mathcal{E}} \\ &\leq C \, \Big\| |u_{(\xi,\varepsilon)} + w|^{\frac{4}{n-2}} \, (u_{(\xi,\varepsilon)} + w) - |u_{(\xi,\varepsilon)} + \tilde{w}|^{\frac{4}{n-2}} \, (u_{(\xi,\varepsilon)} + \tilde{w}) \\ &\qquad - \frac{n+2}{n-2} \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, (w - \tilde{w}) \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ &\leq C \, (\|w\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{4}{n-2}} + \|\tilde{w}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{4}{n-2}}) \, \|w - \tilde{w}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \end{aligned}$$

for all functions $w, \tilde{w} \in \mathcal{E}_{(\xi,\varepsilon)}$. This implies

$$\|\Phi_{(\xi,\varepsilon)}(w) - \Phi_{(\xi,\varepsilon)}(\tilde{w})\|_{\mathcal{E}} \leq C \, (\|w\|_{\mathcal{E}}^{\frac{4}{n-2}} + \|\tilde{w}\|_{\mathcal{E}}^{\frac{4}{n-2}}) \, \|w - \tilde{w}\|_{\mathcal{E}}$$

for $w, \tilde{w} \in \mathcal{E}_{(\xi,\varepsilon)}$. Hence, if α_1 is sufficiently small, then the contraction mapping principle implies that the mapping $\Phi_{(\xi,\varepsilon)}$ has a unique fixed point. From this the assertion follows easily.

We next define a function $\mathcal{F}_g : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{F}_g(\xi,\varepsilon) &= \int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}^2 - (n-2)^2\, |v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} \Big) \\ &\quad - 2(n-2) \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}}.\end{aligned}$$

If we choose α_1 small enough, then we obtain the following result:

Proposition 6. *The function $\mathcal{F}_g$ is continuously differentiable. Moreover, if $(\bar{\xi},\bar{\varepsilon})$ is a critical point of the function $\mathcal{F}_g$, then the function $v_{(\bar{\xi},\bar{\varepsilon})}$ is a non-negative weak solution of the equation*

$$\Delta_g v_{(\bar{\xi},\bar{\varepsilon})} - \frac{n-2}{4(n-1)}\, R_g\, v_{(\bar{\xi},\bar{\varepsilon})} + n(n-2)\, v_{(\bar{\xi},\bar{\varepsilon})}^{\frac{n+2}{n-2}} = 0.$$

Proof. By definition of $v_{(\xi,\varepsilon)}$, we can find real numbers $a_k(\xi,\varepsilon)$, $k = 0,1,\ldots,n$, such that

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)}\, \psi - n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, \psi \Big) \\ &= \sum_{k=0}^n a_k(\xi,\varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \psi\end{aligned}$$

for all test functions $\psi \in \mathcal{E}$. This implies

$$\frac{\partial}{\partial \varepsilon} \mathcal{F}_g(\varepsilon,\xi) = 2 \sum_{k=0}^n a_k(\xi,\varepsilon) \int_{\mathbb{R}^n} \varphi_{\xi,\varepsilon,k)}\, \frac{\partial}{\partial \varepsilon} v_{(\xi,\varepsilon)}$$

and

$$\frac{\partial}{\partial \xi_j} \mathcal{F}_g(\varepsilon,\xi) = 2 \sum_{k=0}^n a_k(\xi,\varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \frac{\partial}{\partial \xi_j} v_{(\xi,\varepsilon)}$$

for $j = 1,\ldots,n$. On the other hand, we have

$$\int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) = 0$$

since $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$. This implies

$$\begin{aligned}0 &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \frac{\partial}{\partial \varepsilon} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \frac{\partial}{\partial \varepsilon} v_{(\xi,\varepsilon)} \\ &\quad + \frac{n-2}{2(n+1)} \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}}\, \varepsilon^{-1}\, \delta_{0k}\end{aligned}$$

and

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \frac{\partial}{\partial \xi_j} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \frac{\partial}{\partial \xi_j} v_{(\xi,\varepsilon)} \\ &- \frac{n-2}{2(n+1)} \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}} \varepsilon^{-1}\, \delta_{jk} \end{aligned}$$

for $j = 1, \ldots, n$. Putting these facts together, we obtain

$$\begin{aligned} &- \frac{n-2}{n+1} \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}} a_0(\xi,\varepsilon) \\ &= \varepsilon\, \frac{\partial}{\partial \varepsilon} \mathcal{F}_g(\xi,\varepsilon) + 2\varepsilon \sum_{k=0}^n a_k(\xi,\varepsilon) \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \end{aligned}$$

and

$$\begin{aligned} &\frac{n-2}{n+1} \Big(\frac{Y(S^n)}{4n(n-1)}\Big)^{\frac{n}{2}} a_j(\xi,\varepsilon) \\ &= \varepsilon\, \frac{\partial}{\partial \xi_j} \mathcal{F}_g(\xi,\varepsilon) + 2\varepsilon \sum_{k=0}^n a_k(\xi,\varepsilon) \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi_{(\xi,\varepsilon,k)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \end{aligned}$$

for $j = 1, \ldots, n$. Hence, if $(\bar{\xi},\bar{\varepsilon})$ is a critical point of $\mathcal{F}_g$, then we have

$$\sum_{k=0}^n |a_k(\bar{\xi},\bar{\varepsilon})| \leq C\, \|v_{(\bar{\xi},\bar{\varepsilon})} - u_{(\bar{\xi},\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \sum_{k=0}^n |a_k(\bar{\xi},\bar{\varepsilon})|,$$

where C is a constant that depends only on n. On the other hand, we have $\|v_{(\bar{\xi},\bar{\varepsilon})} - u_{(\bar{\xi},\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\, \alpha_1$. Hence, if we choose α_1 sufficiently small, then we must have $a_k(\bar{\xi},\bar{\varepsilon}) = 0$ for $k = 0, 1, \ldots, n$. Thus, we conclude that

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\bar{\xi},\bar{\varepsilon})}, d\psi\rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\bar{\xi},\bar{\varepsilon})}\, \psi - n(n-2)\, |v_{(\bar{\xi},\bar{\varepsilon})}|^{\frac{4}{n-2}}\, v_{(\bar{\xi},\bar{\varepsilon})}\, \psi\Big) = 0$$

for all test functions $\psi \in \mathcal{E}$. It remains to show that the function $v_{(\bar{\xi},\bar{\varepsilon})}$ is non-negative. To that end, we put $\psi = \min\{v_{(\bar{\xi},\bar{\varepsilon})}, 0\}$. Since $v_{(\bar{\xi},\bar{\varepsilon})} \in \mathcal{E}$, we conclude that $\psi \in \mathcal{E}$. This implies

$$\begin{aligned} &\int_{\{v_{(\bar{\xi},\bar{\varepsilon})}<0\}} \Big(|dv_{(\bar{\xi},\bar{\varepsilon})}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, v_{(\bar{\xi},\bar{\varepsilon})}^2\Big) \\ &= n(n-2) \int_{\{v_{(\bar{\xi},\bar{\varepsilon})}<0\}} |v_{(\bar{\xi},\bar{\varepsilon})}|^{\frac{2n}{n-2}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned}&\bigg(\int_{\{v_{(\bar{\xi},\bar{\varepsilon})}<0\}} |v_{(\bar{\xi},\bar{\varepsilon})}|^{\frac{2n}{n-2}}\bigg)^{\frac{n-2}{n}}\\ &\leq 2K \int_{\{v_{(\bar{\xi},\bar{\varepsilon})}<0\}} \Big(|dv_{(\bar{\xi},\bar{\varepsilon})}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, v_{(\bar{\xi},\bar{\varepsilon})}^2\Big)\end{aligned}$$

by Corollary 3. From this we deduce that either $v_{(\bar{\xi},\bar{\varepsilon})} \geq 0$ almost everywhere, or

$$\bigg(\int_{\{v_{(\bar{\xi},\bar{\varepsilon})}<0\}} |v_{(\bar{\xi},\bar{\varepsilon})}|^{\frac{2n}{n-2}}\bigg)^{\frac{2}{n}} \geq \frac{1}{2n(n-2)\,K}.$$

On the other hand, we have

$$\bigg(\int_{\{v_{(\bar{\xi},\bar{\varepsilon})}<0\}} |v_{(\bar{\xi},\bar{\varepsilon})}|^{\frac{2n}{n-2}}\bigg)^{\frac{n-2}{2n}} \leq \bigg(\int_{\mathbb{R}^n} |v_{(\bar{\xi},\bar{\varepsilon})} - u_{(\bar{\xi},\bar{\varepsilon})}|^{\frac{2n}{n-2}}\bigg)^{\frac{n-2}{2n}} \leq C\,\alpha_1.$$

Hence, if α_1 is sufficiently small, then we have $v_{(\bar{\xi},\bar{\varepsilon})} \geq 0$ almost everywhere.

3. An estimate for the energy of a "bubble"

Throughout this paper, we fix a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. We assume that W_{ijkl} satisfy all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of W are non-zero, so that

$$\sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 > 0.$$

For abbreviation, we put

$$H_{ik}(x) = \sum_{p,q=1}^n W_{ipkq}\, x_p\, x_q$$

and

$$\overline{H}_{ik}(x) = (1 - |x|^2)\, H_{ik}(x).$$

It is easy to see that $H_{ik}(x)$ is trace-free, $\sum_{i=1}^n x_i\, H_{ik}(x) = 0$, and $\sum_{i=1}^n \partial_i H_{ik}(x) = 0$ for all $x \in \mathbb{R}^n$.

We consider a Riemannian metric of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ satisfying $h(x) = 0$ for $|x| \geq 1$,

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$$

for all $x \in \mathbb{R}^n$, and

$$h_{ik}(x) = \mu\,(\lambda^2 - |x|^2)\, H_{ik}(x)$$

for $|x| \leq \rho$. We assume that the parameters λ, μ, and ρ are chosen such that $\mu \leq 1$ and $\lambda \leq \rho \leq 1$. Note that $\sum_{i=1}^n x_i\, h_{ik}(x) = 0$ and $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$.

Given any pair $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$, there exists a unique function $v_{(\xi,\varepsilon)}$ such that $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ and

$$\int_{\mathbb{R}^n} \Big(\langle dv_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, \psi - n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, \psi \Big) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$ (see Proposition 5). For abbreviation, let

$$\Omega = \Big\{ (\xi,\varepsilon) \in \mathbb{R}^n \times \mathbb{R} : |\xi| < 1,\ \frac{n-8}{3(n+4)} < \varepsilon^2 < \frac{2(n-8)}{3(n+4)} \Big\}.$$

Proposition 7. *For every pair $(\xi,\varepsilon) \in \lambda\,\Omega$, we have*

$$\begin{aligned} &\Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ &\leq C\, \lambda^4\, \mu + C \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}} \end{aligned}$$

and

$$\begin{aligned} &\Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \\ &\qquad + \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ &\leq C\, \lambda^8\, \mu^2 + C \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}}. \end{aligned}$$

Proof. For abbreviation, we define two functions A_1 and A_2 by

$$A_1 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}$$

and

$$A_2 = \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}.$$

Using Proposition 26 and the identity $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$, we obtain

$$|R_g(x)| \leq C\, |h(x)|^2\, |\partial^2 h(x)| + C\, |\partial h(x)|^2 \leq C\, \mu^2\, (\lambda + |x|)^6$$

for $|x| \leq \rho$. This implies

$$\begin{aligned} |A_1| &= \Big| \sum_{i,k=1}^n \partial_i \big[(g^{ik} - \delta_{ik})\, \partial_k u_{(\xi,\varepsilon)} \big] - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} \Big| \\ &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{4-n} \end{aligned}$$

and

$$\begin{aligned}|A_1 + A_2| &= \Big| \sum_{i,k=1}^n \partial_i \big[(g^{ik} - \delta_{ik} + h_{ik}) \, \partial_k u_{(\xi,\varepsilon)} \big] - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} \Big| \\ &\leq C \, \lambda^{\frac{n-2}{2}} \, \mu^2 \, (\lambda + |x|)^{8-n}\end{aligned}$$

for $|x| \leq \rho$. Hence, we obtain

$$\|A_1\|_{L^{\frac{2n}{n+2}}(B_\rho(0))} \leq C \, \lambda^{\frac{n-2}{2}} \, \mu \, \bigg(\int_{\mathbb{R}^n} (\lambda + |x|)^{-\frac{2n(n-4)}{n+2}} \bigg)^{\frac{n+2}{2n}} \leq C \, \lambda^4 \, \mu$$

and

$$\|A_1 + A_2\|_{L^{\frac{2n}{n+2}}(B_\rho(0))} \leq C \, \lambda^{\frac{n-2}{2}} \, \mu^2 \, \bigg(\int_{\mathbb{R}^n} (\lambda + |x|)^{-\frac{2n(n-8)}{n+2}} \bigg)^{\frac{n+2}{2n}} \leq C \, \lambda^8 \, \mu^2.$$

On the other hand, we have

$$|A_1(x)| \leq C \, \lambda^{\frac{n-2}{2}} \, |x|^{-n}$$

for $\rho \leq |x| \leq 1$ and

$$|A_2(x)| \leq C \, \lambda^{\frac{n-2}{2}} \, \mu \, |x|^{4-n}$$

for $|x| \geq \rho$. Since the function $A_1(x)$ vanishes for $|x| \geq 1$, we conclude that

$$\|A_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\rho(0))} \leq C \, \lambda^{\frac{n-2}{2}} \, \bigg(\int_{\mathbb{R}^n \setminus B_\rho(0)} |x|^{-\frac{2n^2}{n+2}} \bigg)^{\frac{n+2}{2n}} \leq C \, \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}}$$

and

$$\|A_2\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\rho(0))} \leq C \, \lambda^{\frac{n-2}{2}} \, \mu \, \bigg(\int_{\mathbb{R}^n \setminus B_\rho(0)} |x|^{-\frac{2n(n-4)}{n+2}} \bigg)^{\frac{n+2}{2n}} \leq C \, \rho^4 \, \mu \, \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}}.$$

Putting these facts together, the assertion follows.

Corollary 8. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \, \lambda^4 \, \mu + C \, \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}}$$

for $(\xi,\varepsilon) \in \lambda \, \Omega$.

Proof. It follows from Proposition 5 that

$$\begin{aligned}&\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ &\leq C \, \Big\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \, R_g \, u_{(\xi,\varepsilon)} + n(n-2) \, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},\end{aligned}$$

where C is a constant that depends only on n. Hence, the assertion follows from Proposition 7.

We now prove a more refined estimate for the difference $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$. Using Proposition 4 with $h = 0$, we conclude that there exists a unique function $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w_{(\xi,\varepsilon)}\, \psi \Big) \\
&= -\int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, \psi
\end{aligned} \tag{3}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 9. *The function $w_{(\xi,\varepsilon)}$ is smooth. Moreover, if $(\xi,\varepsilon) \in \lambda\, \Omega$, then we have*

$$\begin{aligned}
|w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{6-n} \\
|\partial w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{5-n} \\
|\partial^2 w_{(\xi,\varepsilon)}(x)| &\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{4-n}
\end{aligned}$$

for all $x \in \mathbb{R}^n$.

Proof. Let $\varphi_{(\xi,\varepsilon,k)}$ be the functions defined in Section 2. We can find real numbers $b_k(\xi,\varepsilon)$, $k = 0, 1, \ldots, n$, such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w_{(\xi,\varepsilon)}\, \psi \Big) \\
&= -\int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, \psi + \sum_{k=0}^n b_k(\xi,\varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi,\varepsilon,k)}\, \psi
\end{aligned}$$

for all test functions $\psi \in \mathcal{E}$. It follows from standard elliptic regularity theory that $w_{(\xi,\varepsilon)}$ is smooth.

In the next step, we establish quantitative estimates for $w_{(\xi,\varepsilon)}$. To that end, we consider a pair $(\xi,\varepsilon) \in \lambda\, \Omega$. A straightforward calculation yields

$$\Big\| \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)} \Big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\, \lambda^4\, \mu. \tag{4}$$

From this we deduce that $\|w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\, \lambda^4\, \mu$ and $\sum_{k=0}^n |b_k(\xi,\varepsilon)| \leq C\, \lambda^4\, \mu$. This implies

$$\begin{aligned}
&\big| \Delta w_{(\xi,\varepsilon)} + n(n+2)\, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\, w_{(\xi,\varepsilon)} \big| \\
&= \Big| \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)} - \sum_{k=0}^n b_k(\xi,\varepsilon)\, \varphi_{(\xi,\varepsilon,k)} \Big| \\
&\leq C\, \lambda^{\frac{n-2}{2}}\, \mu\, (\lambda + |x|)^{4-n}
\end{aligned}$$

for all $x \in \mathbb{R}^n$. We claim that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x)| \leq C \, \lambda^4 \, \mu.$$

To show this, we fix a point $x_0 \in \mathbb{R}^n$ and put $r = \frac{1}{2} \, (\lambda + |x_0|)$. Clearly, $\lambda + |x| \geq r$ for all $x \in B_r(x_0)$. This implies

$$u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} \leq C \, r^{-2}$$

and

$$\big|\Delta w_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)}\big| \leq C \, \lambda^{\frac{n-2}{2}} \, \mu \, r^{4-n}$$

for all $x \in B_r(x_0)$. Using standard interior estimates, we obtain

$$\begin{aligned} r^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x_0)| &\leq C \, \|w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(B_r(x_0))} \\ &\quad + C \, r^{\frac{n+2}{2}} \, \big\|\Delta w_{(\xi,\varepsilon)} + n(n+2) \, u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \, w_{(\xi,\varepsilon)}\big\|_{L^\infty(B_r(x_0))} \\ &\leq C \, \lambda^4 \, \mu + C \, \lambda^{\frac{n-2}{2}} \, \mu \, r^{-\frac{n-10}{2}} \\ &\leq C \, \lambda^4 \, \mu. \end{aligned}$$

Thus, we conclude that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x)| \leq C \, \lambda^4 \, \mu,$$

as claimed. Since $\sup_{x \in \mathbb{R}^n} |x|^{\frac{n-2}{2}} \, |w_{(\xi,\varepsilon)}(x)| < \infty$, we can express the function $w_{(\xi,\varepsilon)}$ in the form

$$w_{(\xi,\varepsilon)}(x) = -\frac{1}{(n-2) \, |S^{n-1}|} \int_{\mathbb{R}^n} |x-y|^{2-n} \, \Delta w_{(\xi,\varepsilon)}(y) \, dy \tag{5}$$

for all $x \in \mathbb{R}^n$.

We can now use a bootstrap argument to prove the desired estimate for $w_{(\xi,\varepsilon)}$. It follows from (5) that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^\beta \, |w_{(\xi,\varepsilon)}(x)| \leq C \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta+2} \, |\Delta w_{(\xi,\varepsilon)}(x)|$$

for all $0 < \beta < n-2$. Since

$$\begin{aligned} |\Delta w_{(\xi,\varepsilon)}(x)| &\leq n(n+2) \, u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} \, |w_{(\xi,\varepsilon)}(x)| \\ &\quad + C \, \lambda^{\frac{n-2}{2}} \, \mu \, (\lambda + |x|)^{4-n} \end{aligned}$$

for all $x \in \mathbb{R}^n$, we conclude that

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^\beta \, |w_{(\xi,\varepsilon)}(x)| &\leq C \, \lambda^2 \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta-2} \, |w_{(\xi,\varepsilon)}(x)| \\ &\quad + C \, \lambda^{\beta - \frac{n-10}{2}} \, \mu \end{aligned}$$

for all $0 < \beta \leq n-6$. Iterating this inequality, we obtain

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{n-6} \, |w_{(\xi,\varepsilon)}(x)| \leq C \, \lambda^{\frac{n-2}{2}} \, \mu.$$

The estimates for the first and second derivatives of $w_{(\xi,\varepsilon)}$ follow now from standard interior estimates.

Corollary 10. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\lambda^{\frac{4(n+2)}{n-2}}\,\mu^{\frac{n+2}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}$$

for $(\xi,\varepsilon) \in \lambda\,\Omega$.

Proof. Consider the functions

$$B_1 = \sum_{i,k=1}^n \partial_i\big[(g^{ik} - \delta_{ik})\,\partial_k w_{(\xi,\varepsilon)}\big] - \frac{n-2}{4(n-1)}\,R_g\,w_{(\xi,\varepsilon)}$$

and

$$B_2 = \sum_{i,k=1}^n \mu\,(\lambda^2 - |x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}.$$

Using (3), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(\langle dw_{(\xi,\varepsilon)}, d\psi\rangle_g + \frac{n-2}{4(n-1)}\,R_g\,w_{(\xi,\varepsilon)}\,\psi - n(n+2)\,u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\,w_{(\xi,\varepsilon)}\,\psi\Big)\\ &= -\int_{\mathbb{R}^n} (B_1 + B_2)\,\psi\end{aligned}$$

for all functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Since $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$, it follows that

$$w_{(\xi,\varepsilon)} = -G_{(\xi,\varepsilon)}(B_1 + B_2).$$

Moreover, we have

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}\big(B_3 + n(n-2)\,B_4\big),$$

where

$$B_3 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)} + n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}$$

and

$$B_4 = |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\,v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}\,u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}\,(v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}).$$

Thus, we conclude that

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}\big(B_1 + B_2 + B_3 + n(n-2)\,B_4\big),$$

where $G_{(\xi,\varepsilon)} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \to \mathcal{E}_{(\xi,\varepsilon)}$ denotes the solution operator constructed in Proposition 4. In particular, we have

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\big\|B_1 + B_2 + B_3 + n(n-2)\,B_4\big\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$$

by Proposition 4. Using Proposition 9, we obtain

$$|B_1(x)| \leq C\,\lambda^{\frac{n-2}{2}}\,\mu^2\,(\lambda + |x|)^{8-n}$$

for $|x| \leq \rho$ and

$$|B_1(x)| \leq C\,\lambda^{\frac{n-2}{2}}\,\mu\,|x|^{4-n}$$

for $\rho \leq |x| \leq 1$. Since the function $B_1(x)$ vanishes for $|x| \geq 1$, we conclude that

$$\|B_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\,\lambda^8\,\mu^2 + C\,\rho^4\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}.$$

Moreover, we have

$$\|B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\,\lambda^8\,\mu^2 + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}}$$

by Proposition 7. Finally, the function B_4 satisfies a pointwise estimate of the form

$$|B_4| \leq C\,|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}|^{\frac{n+2}{n-2}},$$

where C is a constant that depends only on n. Hence, it follows from Corollary 8 that

$$\begin{aligned}\|B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} &\leq C\,\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{n+2}{n-2}} \\ &\leq C\,\lambda^{\frac{4(n+2)}{n-2}}\,\mu^{\frac{n+2}{n-2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n+2}{2}}.\end{aligned}$$

Putting these facts together, we obtain

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\lambda^{\frac{4(n+2)}{n-2}}\,\mu^{\frac{n+2}{n-2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}},$$

as claimed.

Proposition 11. *We have*

$$\begin{aligned}&\bigg|\int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\,R_g\,(v_{(\xi,\varepsilon)}^2 - u_{(\xi,\varepsilon)}^2)\Big) \\ &\quad + \int_{\mathbb{R}^n} n(n-2)\,(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\,u_{(\xi,\varepsilon)}\,v_{(\xi,\varepsilon)} \\ &\quad - \int_{\mathbb{R}^n} n(n-2)\,(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \\ &\quad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\,(\lambda^2 - |x|^2)\,H_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}\,w_{(\xi,\varepsilon)}\bigg| \\ &\leq C\,\lambda^{\frac{8n}{n-2}}\,\mu^{\frac{2n}{n-2}} + C\,\lambda^4\,\mu\,\Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C\,\Big(\frac{\lambda}{\rho}\Big)^{n-2}\end{aligned}$$

for $(\xi,\varepsilon) \in \lambda\,\Omega$.

Proof. Using Proposition 5 with $\psi = v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$, we obtain

$$\int_{\mathbb{R}^n} \Big(|dv_{(\xi,\varepsilon)}|_g^2 - \langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)} \rangle_g + \frac{n-2}{4(n-1)}\, R_g\, v_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \Big) - \int_{\mathbb{R}^n} n(n-2)\, |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}}\, v_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) = 0.$$

Moreover, it follows from Proposition 7 and Corollary 8 that

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} \Big(\langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)} \rangle_g - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \Big) \\
&\quad - \int_{\mathbb{R}^n} n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \\
&\quad - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \bigg| \\
&\leq \bigg\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, R_g\, u_{(\xi,\varepsilon)} + n(n-2)\, u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \\
&\quad + \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)} \bigg\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
&\quad \cdot \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\leq C\, \lambda^{12}\, \mu^3 + C\, \lambda^4\, \mu \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}} + C \Big(\frac{\lambda}{\rho} \Big)^{n-2}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}) \bigg| \\
&\leq C\, \lambda^4\, \mu\, \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\
&\leq C\, \lambda^{\frac{8n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^4\, \mu \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}}
\end{aligned}$$

by (4) and Corollary 10. Putting these facts together, the assertion follows.

Proposition 12. *We have*

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} - \frac{2}{n} \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \bigg| \\
&\leq C\, \lambda^{\frac{8n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C \Big(\frac{\lambda}{\rho} \Big)^n
\end{aligned}$$

for $(\xi, \varepsilon) \in \lambda\, \Omega$.

Proof. We have the pointwise estimate

$$\begin{aligned}&\Big|(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} - \frac{2}{n}\,(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big| \\ &\leq C\,|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}},\end{aligned}$$

where C is a constant that depends only on n. This implies

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}})\, u_{(\xi,\varepsilon)}\, v_{(\xi,\varepsilon)} - \frac{2}{n}\int_{\mathbb{R}^n}(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}})\Big| \\ &\leq C\,\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{2n}{n-2}} \\ &\leq C\,\lambda^{\frac{8n}{n-2}}\,\mu^{\frac{2n}{n-2}} + C\left(\frac{\lambda}{\rho}\right)^n\end{aligned}$$

by Corollary 8.

Proposition 13. *We have*

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\Big(|du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)}\,R_g\,u_{(\xi,\varepsilon)}^2 - n(n-2)\,u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}\Big) \\ &\qquad - \int_{B_\rho(0)}\frac{1}{2}\sum_{i,k,l=1}^n h_{il}\,h_{kl}\,\partial_i u_{(\xi,\varepsilon)}\,\partial_k u_{(\xi,\varepsilon)} \\ &\qquad + \int_{B_\rho(0)}\frac{n-2}{16(n-1)}\sum_{i,k,l=1}^n(\partial_l h_{ik})^2\,u_{(\xi,\varepsilon)}^2\Big| \\ &\leq C\,\lambda^{12}\,\mu^3 + C\left(\frac{\lambda}{\rho}\right)^{n-2}\end{aligned}$$

for all $(\xi,\varepsilon)\in\lambda\,\Omega$.

Proof. Note that

$$\begin{aligned}&\Big|g^{ik}(x) - \delta_{ik} + h_{ik}(x) - \frac{1}{2}\sum_{l=1}^n h_{il}(x)\,h_{kl}(x)\Big| \\ &\leq C\,|h(x)|^3 \leq C\,\mu^3\,(\lambda + |x|)^{12}\end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} \big(|du_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|^2\big) + \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \\
&\qquad - \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il}\, h_{kl}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \bigg| \\
&\leq C\, \lambda^{n-2}\, \mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{14-2n} + C\, \lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{2-2n} \\
&\leq C\, \lambda^{12}\, \mu^3 + C \Big(\frac{\lambda}{\rho}\Big)^{n-2}.
\end{aligned}$$

By Proposition 26, the scalar curvature of g satisfies the estimate

$$\begin{aligned}
&\bigg| R_g(x) + \frac{1}{4} \sum_{i,k,l=1}^n (\partial_l h_{ik}(x))^2 \bigg| \\
&\leq C\, |h(x)|^2\, |\partial^2 h(x)| + C\, |h(x)|\, |\partial h(x)|^2 \\
&\leq C\, \mu^3\, (\lambda + |x|)^{10}
\end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} R_g\, u_{(\xi,\varepsilon)}^2 + \int_{B_\rho(0)} \frac{1}{4} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2\, u_{(\xi,\varepsilon)}^2 \bigg| \\
&\leq C\, \lambda^{12}\, \mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{14-2n} + C\, \lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{4-2n} \\
&\leq C\, \lambda^{12}\, \mu^3 + C\, \rho^2 \Big(\frac{\lambda}{\rho}\Big)^{n-2}.
\end{aligned}$$

At this point, we use the formula

$$\begin{aligned}
&\partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)}\, \partial_i \partial_k (u_{(\xi,\varepsilon)}^2) \\
&= \frac{1}{n} \Big(|du_{(\xi,\varepsilon)}|^2 - \frac{n-2}{4(n-1)}\, \Delta(u_{(\xi,\varepsilon)}^2) \Big)\, \delta_{ik}.
\end{aligned}$$

Since h_{ik} is trace-free, we obtain

$$\sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} = \frac{n-2}{4(n-1)} \sum_{i,k=1}^n h_{ik}\, \partial_i \partial_k (u_{(\xi,\varepsilon)}^2),$$

hence

$$\int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} = \int_{\mathbb{R}^n} \frac{n-2}{4(n-1)} \sum_{i,k=1}^n \partial_i \partial_k h_{ik}\, u_{(\xi,\varepsilon)}^2.$$

Since $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$, it follows that

$$\bigg| \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \bigg| \leq C \int_{\mathbb{R}^n \setminus B_\rho(0)} u_{(\xi,\varepsilon)}^2 \leq C\, \rho^2 \Big(\frac{\lambda}{\rho}\Big)^{n-2}.$$

Putting these facts together, the assertion follows.

Corollary 14. *The function $\mathcal{F}_g(\xi,\varepsilon)$ satisfies the estimate*

$$\begin{aligned}
\bigg| \mathcal{F}_g(\xi,\varepsilon) &- \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il}\, h_{kl}\, \partial_i u_{(\xi,\varepsilon)}\, \partial_k u_{(\xi,\varepsilon)} \\
&+ \int_{B_\rho(0)} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2\, u_{(\xi,\varepsilon)}^2 \\
&- \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu\, (\lambda^2 - |x|^2)\, H_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, w_{(\xi,\varepsilon)} \bigg| \\
&\leq C\, \lambda^{\frac{8n}{n-2}}\, \mu^{\frac{2n}{n-2}} + C\, \lambda^4\, \mu \Big(\frac{\lambda}{\rho}\Big)^{\frac{n-2}{2}} + C \Big(\frac{\lambda}{\rho}\Big)^{n-2}
\end{aligned}$$

for $(\xi,\varepsilon) \in \lambda\, \Omega$.

Proof. This follows by combining Proposition 11, Proposition 12, and Proposition 13.

4. Finding a critical point of an auxiliary function

We define a function $F : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ as follows: given any pair $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$, we define

$$\begin{aligned}
F(\xi,\varepsilon) &= \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n \overline{H}_{il}(x)\, \overline{H}_{kl}(x)\, \partial_i u_{(\xi,\varepsilon)}(x)\, \partial_k u_{(\xi,\varepsilon)}(x) \\
&- \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, u_{(\xi,\varepsilon)}(x)^2 \\
&+ \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}(x)\, \partial_i \partial_k u_{(\xi,\varepsilon)}(x)\, z_{(\xi,\varepsilon)}(x),
\end{aligned}$$

where $z_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ satisfies the relation

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\langle dz_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2)\, u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}}\, z_{(\xi,\varepsilon)}\, \psi \Big) \\
&= - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}\, \partial_i \partial_k u_{(\xi,\varepsilon)}\, \psi
\end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Our goal in this section is to show that the function $F(\xi,\varepsilon)$ has a critical point.

Proposition 15. *The function $F(\xi,\varepsilon)$ satisfies $F(\xi,\varepsilon) = F(-\xi,\varepsilon)$ for all $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,\infty)$. Consequently, we have $\frac{\partial}{\partial \xi_p} F(0,\varepsilon) = 0$ and $\frac{\partial^2}{\partial \varepsilon \, \partial \xi_p} F(0,\varepsilon) = 0$ for all $\varepsilon > 0$ and $p = 1, \ldots, n$.*

Proof. This follows immediately from the relation $\overline{H}_{ik}(-x) = \overline{H}_{ik}(x)$.

Proposition 16. *We have*

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \, x_p \, x_q \\
&= \frac{2}{n(n+2)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+3} \\
&+ \frac{1}{n(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \, r^{n+3}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{i,k=1}^n H_{ik}(x)^2 \, x_p \, x_q \\
&= \frac{2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+5} \\
&+ \frac{1}{2n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \, r^{n+5}.
\end{aligned}$$

Proof. By definition of $H_{ik}(x)$, we have

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \, x_p \, x_q \\
&= \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^n (W_{ijkl} + W_{ilkj}) \, (W_{imkl} + W_{ilkm}) \, x_j \, x_m \, x_p \, x_q \\
&= \frac{2}{n(n+2)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+3} \\
&+ \frac{1}{n(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \, r^{n+3}.
\end{aligned}$$

Moreover, it follows from Corollary 29 that

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{i,k=1}^n H_{ik}(x)^2 \, x_p \, x_q \\
&= \int_{\partial B_r(0)} \sum_{i,j,k,l,m,s=1}^n W_{ijkl} \, W_{imks} \, x_j \, x_l \, x_m \, x_s \, x_p \, x_q \\
&= \frac{2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+5} \\
&+ \frac{1}{2n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \, r^{n+5}.
\end{aligned}$$

This completes the proof.

Proposition 17. *We have*

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \, x_p \, x_q \\
&= \frac{2}{n(n+2)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \\
&\qquad \cdot \Big[r^{n+3} - \frac{2(n+8)}{n+4} \, r^{n+5} + \frac{n+16}{n+4} \, r^{n+7} \Big] \\
&+ \frac{1}{n(n+2)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \\
&\qquad \cdot \Big[r^{n+3} - \frac{2(n+6)}{n+4} \, r^{n+5} + \frac{n+10}{n+4} \, r^{n+7} \Big].
\end{aligned}$$

Proof. Using the identity

$$\partial_l \overline{H}_{ik}(x) = (1 - |x|^2) \, \partial_l H_{ik}(x) - 2 \, H_{ik}(x) \, x_l$$

and Euler's theorem, we obtain

$$\begin{aligned}
&\sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\
&= (1 - |x|^2)^2 \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \\
&- 4\,(1 - |x|^2) \sum_{i,k,l=1}^n H_{ik}(x)\, x_l\, \partial_l H_{ik}(x) + 4\,|x|^2 \sum_{i,k=1}^n H_{ik}(x)^2 \\
&= (1 - |x|^2)^2 \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 - 4\,(2 - 3\,|x|^2) \sum_{i,k=1}^n H_{ik}(x)^2.
\end{aligned}$$

Hence, the assertion follows from the previous proposition.

Corollary 18. *We have*

$$\begin{aligned}
\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 &= \frac{1}{n}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
&\cdot \Big[r^{n+1} - \frac{2(n+4)}{n+2}\, r^{n+3} + \frac{n+8}{n+2}\, r^{n+5}\Big].
\end{aligned}$$

Proposition 19. *We have*

$$\begin{aligned}
F(0, \varepsilon) = &-\frac{(n-2)(n+4)}{16n(n-1)(n+2)}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
&\cdot \Big[\frac{n-8}{n+4}\,\varepsilon^4 - 2\,\varepsilon^6 + \frac{n+8}{n-10}\,\varepsilon^8\Big] \int_0^\infty (1 + r^2)^{2-n}\, r^{n+3}\, dr.
\end{aligned}$$

Proof. Note that $z_{(0,\varepsilon)}(x) = 0$ for all $x \in \mathbb{R}^n$. This implies

$$F(0, \varepsilon) = -\int_{\mathbb{R}^n} \frac{n-2}{16(n-1)}\, \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{2-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2.$$

Using Corollary 18, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{2-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\
&= \frac{1}{n}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\
&\qquad \cdot \int_0^\infty (1 + r^2)^{2-n} \Big[\varepsilon^4\, r^{n+1} - \frac{2(n+4)}{n+2}\, \varepsilon^6\, r^{n+3} + \frac{n+8}{n+2}\, \varepsilon^8\, r^{n+5}\Big]\, dr.
\end{aligned}$$

Moreover, we have

$$\int_0^\infty (1+r^2)^{2-n}\, r^{n+1}\, dr = \frac{n-8}{n+2} \int_0^\infty (1+r^2)^{2-n}\, r^{n+3}\, dr$$

and

$$\int_0^\infty (1+r^2)^{2-n}\, r^{n+5}\, dr = \frac{n+4}{n-10} \int_0^\infty (1+r^2)^{2-n}\, r^{n+3}\, dr$$

by Proposition 27. From this the assertion follows.

Corollary 20. *Assume that $n \geq 52$. Moreover, suppose that $\varepsilon_* > 0$ is defined by*

$$\Big(3 + \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}}\Big)\, \varepsilon_*^2 = \frac{2(n-8)}{n+4}. \tag{6}$$

Then $(0, \varepsilon_*)$ is a critical point of the function $F(\xi, \varepsilon)$. Moreover, we have $\frac{\partial^2}{\partial \varepsilon^2} F(0, \varepsilon_*) > 0$.

In the next step, we show that $(0, \varepsilon_*)$ is a strict local minimum of the function F. To that end, we compute the Hessian of F at a point $(0, \varepsilon)$.

Proposition 21. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned}
\frac{\partial^2}{\partial \xi_p\, \partial \xi_q} F(0, \varepsilon) &= \int_{\mathbb{R}^n} (n-2)^2\, \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{-n} \sum_{l=1}^n \overline{H}_{pl}(x)\, \overline{H}_{ql}(x) \\
&- \int_{\mathbb{R}^n} \frac{(n-2)^2}{4}\, \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, x_p\, x_q \\
&+ \int_{\mathbb{R}^n} \frac{(n-2)^2}{8(n-1)}\, \varepsilon^{n-2}\, (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, \delta_{pq}.
\end{aligned}$$

Proof. Using the identity

$$\begin{aligned}
&\sum_{i,k,l=1}^n \overline{H}_{il}(x)\, \overline{H}_{kl}(x)\, \partial_i u_{(\xi,\varepsilon)}(x)\, \partial_k u_{(\xi,\varepsilon)}(x) \\
&= (n-2)^2\, \varepsilon^{n-2}\, (\varepsilon^2 + |x-\xi|^2)^{-n} \sum_{i,k,l=1}^n \overline{H}_{il}(x)\, \overline{H}_{kl}(x)\, (x_i - \xi_i)\, (x_k - \xi_k) \\
&= (n-2)^2\, \varepsilon^{n-2}\, (\varepsilon^2 + |x-\xi|^2)^{-n} \sum_{i,k,l=1}^n \overline{H}_{il}(x)\, \overline{H}_{kl}(x)\, \xi_i\, \xi_k,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi_p\,\partial\xi_q}\bigg(\frac{1}{2}\sum_{i,k,l=1}^n \overline{H}_{il}(x)\,\overline{H}_{kl}(x)\,\partial_i u_{(\xi,\varepsilon)}(x)\,\partial_k u_{(\xi,\varepsilon)}(x)\bigg)\bigg|_{\xi=0} \\
&= (n-2)^2\,\varepsilon^{n-2}\,(\varepsilon^2+|x|^2)^{-n}\sum_{l=1}^n \overline{H}_{pl}(x)\,\overline{H}_{ql}(x).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi_p\,\partial\xi_q}\bigg(\frac{n-2}{16(n-1)}\sum_{i,k,l=1}^n (\partial_l\overline{H}_{ik}(x))^2\,u_{(\xi,\varepsilon)}(x)^2\bigg)\bigg|_{\xi=0} \\
&= \frac{(n-2)^2}{4}\,\varepsilon^{n-2}\,(\varepsilon^2+|x|^2)^{-n}\sum_{i,k,l=1}^n (\partial_l\overline{H}_{ik}(x))^2\,x_p\,x_q \\
&\quad - \frac{(n-2)^2}{8(n-1)}\,\varepsilon^{n-2}\,(\varepsilon^2+|x|^2)^{1-n}\sum_{i,k,l=1}^n (\partial_l\overline{H}_{ik}(x))^2\,\delta_{pq}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\sum_{i,k=1}^n \overline{H}_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}(x) \\
&= n(n-2)\,\varepsilon^{\frac{n-2}{2}}\,(\varepsilon^2+|x-\xi|^2)^{-\frac{n+2}{2}}\sum_{i,k=1}^n \overline{H}_{ik}(x)\,(x_i-\xi_i)\,(x_k-\xi_k) \\
&= n(n-2)\,\varepsilon^{\frac{n-2}{2}}\,(\varepsilon^2+|x-\xi|^2)^{-\frac{n+2}{2}}\sum_{i,k=1}^n \overline{H}_{ik}(x)\,\xi_i\,\xi_k
\end{aligned}$$

since $\overline{H}_{ik}(x)$ is trace-free. Thus, we conclude that

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi_p\,\partial\xi_q}\bigg(\sum_{i,k=1}^n \overline{H}_{ik}(x)\,\partial_i\partial_k u_{(\xi,\varepsilon)}(x)\,z_{(\xi,\varepsilon)}(x)\bigg)\bigg|_{\xi=0} \\
&= 2n(n-2)\,\varepsilon^{\frac{n-2}{2}}\,(\varepsilon^2+|x|^2)^{-\frac{n+2}{2}}\sum_{i,k=1}^n \overline{H}_{pq}(x)\,z_{(0,\varepsilon)}(x) = 0.
\end{aligned}$$

From this the assertion follows.

Proposition 22. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned}
&\frac{\partial^2}{\partial \xi_p \, \partial \xi_q} F(0, \varepsilon) \\
&= \frac{4(n-2)^2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \\
&\quad \cdot \Big[\varepsilon^4 - \frac{3(n+6)}{2(n-8)} \, \varepsilon^6 \Big] \int_0^\infty (1 + r^2)^{-n} \, r^{n+5} \, dr \\
&+ \frac{(n-2)^2}{n(n+2)(n+4)} \, |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \, \delta_{pq} \\
&\quad \cdot \Big[\varepsilon^4 - \frac{n+7}{n-8} \, \varepsilon^6 \Big] \int_0^\infty (1 + r^2)^{-n} \, r^{n+5} \, dr.
\end{aligned}$$

Proof. Using the identity

$$\begin{aligned}
&\int_{\partial B_r(0)} \sum_{l=1}^n \overline{H}_{pl}(x) \, \overline{H}_{ql}(x) \\
&= \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^n W_{ipkl} \, W_{jqml} \, x_i \, x_j \, x_k \, x_m \, (1 - |x|^2)^2 \\
&= \frac{1}{n(n+2)} \, |S^{n-1}| \\
&\quad \cdot \sum_{i,j,k,l,m=1}^n W_{ipkl} \, W_{jqml} \, (\delta_{ij} \, \delta_{km} + \delta_{ik} \, \delta_{jm} + \delta_{im} \, \delta_{jk}) \, r^{n+3} \, (1 - r^2)^2 \\
&= \frac{1}{2n(n+2)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \, r^{n+3} \, (1 - r^2)^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2} \, (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n \overline{H}_{pl}(x) \, \overline{H}_{ql}(x) \\
&= \frac{1}{2n(n+2)} \, |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) \, (W_{iqkl} + W_{ilkq}) \\
&\quad \cdot \int_0^\infty (1 + r^2)^{-n} \Big[\varepsilon^2 \, r^{n+3} - 2 \, \varepsilon^4 \, r^{n+5} + \varepsilon^6 \, r^{n+7} \Big] \, dr.
\end{aligned}$$

Similarly, it follows from Proposition 17 that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2}\,(\varepsilon^2+|x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\, x_p\, x_q \\
&= \frac{2}{n(n+2)}\,|S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl}+W_{ilkp})\,(W_{iqkl}+W_{ilkq}) \\
&\qquad \cdot \int_0^\infty (1+r^2)^{-n}\,\Big[\varepsilon^2\,r^{n+3} - \frac{2(n+8)}{n+4}\,\varepsilon^4\,r^{n+5} + \frac{n+16}{n+4}\,\varepsilon^6\,r^{n+7}\Big]\,dr \\
&+ \frac{1}{n(n+2)}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl}+W_{ilkj})^2\,\delta_{pq} \\
&\qquad \cdot \int_0^\infty (1+r^2)^{-n}\,\Big[\varepsilon^2\,r^{n+3} - \frac{2(n+6)}{n+4}\,\varepsilon^4\,r^{n+5} + \frac{n+10}{n+4}\,\varepsilon^6\,r^{n+7}\Big]\,dr.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon^{n-2}\,(\varepsilon^2+|x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2\,\delta_{pq} \\
&= \frac{1}{n}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl}+W_{ilkj})^2\,\delta_{pq} \\
&\qquad \cdot \int_0^\infty (1+r^2)^{1-n}\,\Big[\varepsilon^2\,r^{n+1} - \frac{2(n+4)}{n+2}\,\varepsilon^4\,r^{n+3} + \frac{n+8}{n+2}\,\varepsilon^6\,r^{n+5}\Big]\,dr.
\end{aligned}$$

by Corollary 18. Using Proposition 21 and the identity

$$\int_0^\infty (1+r^2)^{1-n}\,r^{n+1}\,dr = \frac{2(n-1)}{n+2}\int_0^\infty (1+r^2)^{-n}\,r^{n+3}\,dr,$$

we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi_p\,\partial\xi_q} F(0,\varepsilon) \\
&= \frac{4(n-2)^2}{n(n+2)(n+4)}\,|S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl}+W_{ilkp})\,(W_{iqkl}+W_{ilkq}) \\
&\qquad \cdot \int_0^\infty (1+r^2)^{-n}\,\Big[\varepsilon^4\,r^{n+5} - \frac{3}{2}\,\varepsilon^6\,r^{n+7}\Big]\,dr \\
&+ \frac{(n-2)^2}{4n(n+2)}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl}+W_{ilkj})^2\,\delta_{pq} \\
&\qquad \cdot \int_0^\infty (1+r^2)^{-n}\,\Big[\frac{2(n+6)}{n+4}\,\varepsilon^4\,r^{n+5} - \frac{n+10}{n+4}\,\varepsilon^6\,r^{n+7}\Big]\,dr \\
&- \frac{(n-2)^2}{8n(n-1)}\,|S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl}+W_{ilkj})^2\,\delta_{pq} \\
&\qquad \cdot \int_0^\infty (1+r^2)^{1-n}\,\Big[\frac{2(n+4)}{n+2}\,\varepsilon^4\,r^{n+3} - \frac{n+8}{n+2}\,\varepsilon^6\,r^{n+5}\Big]\,dr.
\end{aligned}$$

Hence, the assertion follows from the identities

$$\begin{aligned}
\int_0^\infty (1+r^2)^{-n}\,r^{n+7}\,dr &= \frac{n+6}{n-8}\int_0^\infty (1+r^2)^{-n}\,r^{n+5}\,dr \\
\int_0^\infty (1+r^2)^{1-n}\,r^{n+3}\,dr &= \frac{2(n-1)}{n+4}\int_0^\infty (1+r^2)^{-n}\,r^{n+5}\,dr \\
\int_0^\infty (1+r^2)^{1-n}\,r^{n+5}\,dr &= \frac{2(n-1)}{n-8}\int_0^\infty (1+r^2)^{-n}\,r^{n+5}\,dr.
\end{aligned}$$

Corollary 23. *Assume that $n \geq 52$ and $\varepsilon_* > 0$ is defined by (6). Then the function $F(\xi,\varepsilon)$ has a strict local minimum at the point $(0,\varepsilon_*)$.*

Proof. It follows from Corollary 20 that $(0,\varepsilon_*)$ is a critical point of the function $F(\xi,\varepsilon)$. Moreover, we have $\frac{\partial^2}{\partial\varepsilon^2}F(0,\varepsilon_*) > 0$. Since $n \geq 52$, we have

$$\frac{6}{n+4} < \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}}.$$

This implies

$$\frac{3(n+6)}{n+4}\,\varepsilon_*^2 < \bigg(3 + \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}}\bigg)\,\varepsilon_*^2 = \frac{2(n-8)}{n+4}.$$

Thus, we conclude that

$$\frac{n+7}{n-8}\,\varepsilon_*^2 < \frac{3(n+6)}{2(n-8)}\,\varepsilon_*^2 < 1.$$

Hence, it follows from Proposition 22 that the matrix $\frac{\partial^2}{\partial \xi_p \, \partial \xi_q} F(0, \varepsilon_*)$ is positive definite. This proves the assertion.

5. Proof of the main theorem

Proposition 24. *Assume that $n \geq 52$. Moreover, let g be a smooth metric on $\mathbb{R}^n$ of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^n$ such that $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1$ for all $x \in \mathbb{R}^n$, $h(x) = 0$ for $|x| \geq 1$, and $h_{ik}(x) = \mu \, (\lambda^2 - |x|^2) \, H_{ik}(x)$ for $|x| \leq \rho$. As above, we assume that $\lambda \leq \rho \leq 1$ and $\mu \leq 1$. If α and $\rho^{2-n} \, \mu^{-2} \, \lambda^{n-10}$ are sufficiently small, then there exists a positive function v such that*

$$\Delta_g v - \frac{n-2}{4(n-1)} \, R_g \, v + n(n-2) \, v^{\frac{n+2}{n-2}} = 0,$$

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} < \Big(\frac{Y(S^n)}{4n(n-1)} \Big)^{\frac{n}{2}},$$

and $\sup_{|x| \leq \lambda} v(x) \geq c \, \lambda^{\frac{2-n}{2}}$. Here, c is a positive constant that depends only on n.

Proof. By Corollary 23, the function $F(\xi, \varepsilon)$ has a strict local minimum at $(0, \varepsilon_*)$. Hence, we can find an open set $\Omega' \subset \Omega$ such that $(0, \varepsilon_*) \in \Omega'$ and

$$F(0, \varepsilon_*) < \inf_{(\xi, \varepsilon) \in \partial \Omega'} F(\xi, \varepsilon) < 0.$$

Using Corollary 14, we obtain

$$\begin{aligned} &|\mathcal{F}_g(\lambda \xi, \lambda \varepsilon) - \lambda^8 \, \mu^2 \, F(\xi, \varepsilon)| \\ &\leq C \, \lambda^{\frac{8n}{n-2}} \, \mu^{\frac{2n}{n-2}} + C \, \lambda^4 \, \mu \, \Big(\frac{\lambda}{\rho} \Big)^{\frac{n-2}{2}} + C \, \Big(\frac{\lambda}{\rho} \Big)^{n-2} \end{aligned}$$

for all $(\xi, \varepsilon) \in \Omega$. This implies

$$\begin{aligned} &|\lambda^{-8} \, \mu^{-2} \, \mathcal{F}_g(\lambda \xi, \lambda \varepsilon) - F(\xi, \varepsilon)| \\ &\leq C \, \lambda^{\frac{16}{n-2}} \, \mu^{\frac{4}{n-2}} + C \, \rho^{\frac{2-n}{2}} \, \mu^{-1} \, \lambda^{\frac{n-10}{2}} + C \, \rho^{2-n} \, \mu^{-2} \, \lambda^{n-10} \end{aligned}$$

for all $(\xi, \varepsilon) \in \Omega$. Hence, if $\rho^{2-n} \, \mu^{-2} \, \lambda^{n-10}$ is sufficiently small, then we have

$$\mathcal{F}_g(0, \lambda \varepsilon_*) < \inf_{(\xi, \varepsilon) \in \partial \Omega'} \mathcal{F}_g(\lambda \xi, \lambda \varepsilon) < 0.$$

Consequently, there exists a point $(\bar{\xi}, \bar{\varepsilon}) \in \Omega'$ such that

$$\mathcal{F}_g(\lambda \bar{\xi}, \lambda \bar{\varepsilon}) = \inf_{(\xi, \varepsilon) \in \Omega'} \mathcal{F}_g(\lambda \xi, \lambda \varepsilon) < 0.$$

By Proposition 6, the function $v = v_{(\lambda \bar{\xi}, \lambda \bar{\varepsilon})}$ is a non-negative weak solution of the partial differential equation

$$\Delta_g v - \frac{n-2}{4(n-1)} \, R_g \, v + n(n-2) \, v^{\frac{n+2}{n-2}} = 0.$$

Using a result of N. Trudinger, we conclude that v is smooth (see [20], Theorem 3 on p. 271). Moreover, we have

$$\begin{aligned} 2(n-2)\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} &= 2(n-2)\left(\frac{Y(S^n)}{4n(n-1)}\right)^{\frac{n}{2}} + \mathcal{F}_g(\lambda\bar{\xi},\lambda\bar{\varepsilon}) \\ &< 2(n-2)\left(\frac{Y(S^n)}{4n(n-1)}\right)^{\frac{n}{2}}. \end{aligned}$$

Finally, it follows from Proposition 5 that $\|v-u_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\,\alpha$. This implies

$$|B_\lambda(0)|^{\frac{n-2}{2n}} \sup_{|x|\leq\lambda} v(x) \geq \|v\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} \geq \|u_{(\lambda\bar{\xi},\lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} - C\,\alpha.$$

Hence, if α is sufficiently small, then we obtain $\lambda^{\frac{n-2}{2}} \sup_{|x|\leq\lambda} v(x) \geq c$.

Proposition 25. *Let $n \geq 52$. Then there exists a smooth metric g on $\mathbb{R}^n$ with the following properties:*

(i) $g_{ik}(x) = \delta_{ik}$ *for* $|x| \geq \frac{1}{2}$

(ii) g *is not conformally flat*

(iii) *There exists a sequence of non-negative smooth functions v_ν ($\nu \in \mathbb{N}$) such that*

$$\Delta_g v_\nu - \frac{n-2}{4(n-1)}\,R_g\,v_\nu + n(n-2)\,v_\nu^{\frac{n+2}{n-2}} = 0$$

for all $\nu \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} v_\nu^{\frac{2n}{n-2}} < \left(\frac{Y(S^n)}{4n(n-1)}\right)^{\frac{n}{2}}$$

for all $\nu \in \mathbb{N}$, and $\sup_{|x|\leq 1} v_\nu(x) \to \infty$ as $\nu \to \infty$.

Proof. Choose a smooth cutoff function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. We define a trace-free symmetric two-tensor on $\mathbb{R}^n$ by

$$h_{ik}(x) = \sum_{N=N_0}^{\infty} \eta(4N^2\,|x-y_N|)\,2^{-N}\,(2^{-N} - |x-y_N|^2)\,H_{ik}(x-y_N),$$

where $y_N = (\frac{1}{N}, 0, \ldots, 0) \in \mathbb{R}^n$. It is straightforward to verify that $h(x)$ is C^∞ smooth.

Let α be the constant appearing in Proposition 24. If N_0 is sufficiently large, then we have $|h(x)|+|\partial h(x)|+|\partial^2 h(x)| \leq \alpha$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq \frac{1}{2}$. Moreover, we have $h_{ik}(x) = 2^{-N}\,(2^{-N} - |x-y_N|^2)\,H_{ik}(x-y_N)$ provided that $N \geq N_0$ and $|x-y_N| \leq \frac{1}{4N^2}$. Hence, we can apply Proposition 24 with $\lambda = 2^{-N/2}$, $\mu = 2^{-N}$, and $\rho = \frac{1}{4N^2}$. From this the assertion follows.

Appendix A. An asymptotic expansion for the scalar curvature

Suppose that $h(x)$ is a trace-free symmetric two-tensor defined on $\mathbb{R}^n$ satisfying $|h(x)| \leq 1$ for all $x \in \mathbb{R}^n$. We define a Riemannian metric g on $\mathbb{R}^n$ by $g(x) = \exp(h(x))$. In this section, we derive an approximate formula for the scalar curvature of this metric. A similar formula is derived in [2].

Proposition 26. *Let R_g be the scalar curvature of g. There exists a constant C, depending only on n, such that*

$$\begin{aligned} &\Big| R_g - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \, \partial_k h_{kl}) - \frac{1}{2} \, \partial_i h_{il} \, \partial_k h_{kl} + \frac{1}{4} \, \partial_l h_{ik} \, \partial_l h_{ik} \Big| \\ &\leq C \, |h|^2 \, |\partial^2 h| + C \, |h| \, |\partial h|^2. \end{aligned}$$

Proof. The Riemann curvature tensor is defined as

$$\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^l \, \Gamma_{il}^m - \Gamma_{ik}^l \, \Gamma_{jl}^m.$$

Hence, the scalar curvature of g is given by

$$R_g = g^{jk} \, (\partial_i \Gamma_{jk}^i - \partial_j \Gamma_{ik}^i + \Gamma_{jk}^l \, \Gamma_{il}^i - \Gamma_{ik}^l \, \Gamma_{jl}^i).$$

Since h is trace-free, we have $\det g(x) = 1$ for all $x \in \mathbb{R}^n$. This implies $\Gamma_{ik}^i = \frac{1}{2} \, g^{il} \, \partial_k g_{il} = \frac{1}{2} \, \partial_k \log \det g = 0$. Therefore, we obtain

$$\begin{aligned} R_g &= g^{jk} \, \partial_i \Gamma_{jk}^i - g^{jk} \, \Gamma_{ik}^l \, \Gamma_{jl}^i \\ &= \partial_i (g^{jk} \, \Gamma_{jk}^i) + g^{jk} \, \Gamma_{ik}^l \, \Gamma_{jl}^i. \end{aligned}$$

Note that

$$g^{jk} \, \Gamma_{jk}^i = g^{il} \, g^{jk} \, \partial_k g_{jl}.$$

From this it follows that

$$\begin{aligned} &\Big| \partial_i (g^{jk} \, \Gamma_{jk}^i) - \partial_i \partial_k h_{ik} + \frac{1}{2} \, \partial_i (h_{il} \, \partial_k h_{kl}) + \frac{1}{2} \, \partial_i (h_{kl} \, \partial_k h_{il}) \Big| \\ &\leq C \, |h|^2 \, |\partial^2 h| + C \, |h| \, |\partial h|^2, \end{aligned}$$

hence

$$\begin{aligned} &\Big| \partial_i (g^{jk} \, \Gamma_{jk}^i) - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \, \partial_k h_{kl}) - \frac{1}{2} \, \partial_i h_{il} \, \partial_k h_{kl} + \frac{1}{2} \, \partial_i h_{kl} \, \partial_k h_{il} \Big| \\ &\leq C \, |h|^2 \, |\partial^2 h| + C \, |h| \, |\partial h|^2, \end{aligned}$$

Moreover, we have

$$\Big| g^{jk} \, \Gamma_{ik}^l \, \Gamma_{jl}^i + \frac{1}{4} \, \partial_l h_{ik} \, \partial_l h_{ik} - \frac{1}{2} \, \partial_i h_{kl} \, \partial_k h_{il} \Big| \leq C \, |h| \, |\partial h|^2.$$

Putting these facts together, we obtain

$$\begin{aligned} &\Big| R_g - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \, \partial_k h_{kl}) - \frac{1}{2} \, \partial_i h_{il} \, \partial_k h_{kl} + \frac{1}{4} \, \partial_l h_{ik} \, \partial_l h_{ik} \Big| \\ &\leq C \, |h|^2 \, |\partial^2 h| + C \, |h| \, |\partial h|^2. \end{aligned}$$

This completes the proof.

Appendix B. Some useful identities

Proposition 27. *Suppose that α and β are real numbers satisfying $2\alpha - 2 > \beta + 1 > 0$. Then*

$$\int_0^\infty (1 + r^2)^{1-\alpha} \, r^\beta \, dr = \frac{2\alpha - 2}{2\alpha - \beta - 3} \int_0^\infty (1 + r^2)^{-\alpha} \, r^\beta \, dr$$

and

$$\int_0^\infty (1 + r^2)^{-\alpha} \, r^{\beta+2} \, dr = \frac{\beta + 1}{2\alpha - \beta - 3} \int_0^\infty (1 + r^2)^{-\alpha} \, r^\beta \, dr.$$

Proof. Using the fundamental theorem of calculus, we obtain

$$\begin{aligned} 0 &= \int_0^\infty \frac{d}{dr} \big[(1 + r^2)^{1-\alpha} \, r^{\beta+1} \big] \, dr \\ &= (\beta + 1) \int_0^\infty (1 + r^2)^{1-\alpha} \, r^\beta \, dr - (2\alpha - 2) \int_0^\infty (1 + r^2)^{-\alpha} \, r^{\beta+2} \, dr. \end{aligned}$$

From this the assertion follows.

Proposition 28. *Suppose that $p(x)$ is a homogenous polynomial of degree d. Then*

$$\int_{\partial B_1(0)} p(x) = \frac{1}{d(n + d - 2)} \int_{\partial B_1(0)} \Delta p(x).$$

Proof. Using the divergence theorem, we obtain

$$\begin{aligned} \int_{\partial B_1(0)} \Delta p(x) &= (n + d - 2) \int_{B_1(0)} \Delta p(x) \\ &= (n + d - 2) \int_{\partial B_1(0)} \sum_{k=1}^n x_k \, \partial_k p(x) \\ &= d(n + d - 2) \int_{\partial B_1(0)} p(x). \end{aligned}$$

Corollary 29. *We have*

$$\int_{\partial B_1(0)} x_i \, x_j = \frac{1}{n} \, |S^{n-1}| \, \delta_{ij},$$

$$\int_{\partial B_1(0)} x_i \, x_j \, x_k \, x_l = \frac{1}{n(n + 2)} \, |S^{n-1}| \, (\delta_{ij} \, \delta_{kl} + \delta_{ik} \, \delta_{jl} + \delta_{il} \, \delta_{jk}),$$

and

$$\begin{aligned}\int_{\partial B_1(0)} x_i\, x_j\, x_k\, x_l\, x_p\, x_q \\ = \frac{1}{n(n+2)(n+4)}\, |S^{n-1}|\, (&\delta_{ij}\,\delta_{kl}\,\delta_{pq} + \delta_{ij}\,\delta_{kp}\,\delta_{lq} + \delta_{ij}\,\delta_{kq}\,\delta_{lp} \\ &+ \delta_{ik}\,\delta_{jl}\,\delta_{pq} + \delta_{ik}\,\delta_{jp}\,\delta_{lq} + \delta_{ik}\,\delta_{jq}\,\delta_{lp} \\ &+ \delta_{il}\,\delta_{jk}\,\delta_{pq} + \delta_{il}\,\delta_{jp}\,\delta_{kq} + \delta_{il}\,\delta_{jq}\,\delta_{kp} \\ &+ \delta_{ip}\,\delta_{jk}\,\delta_{lq} + \delta_{ip}\,\delta_{jl}\,\delta_{kq} + \delta_{ip}\,\delta_{jq}\,\delta_{kl} \\ &+ \delta_{iq}\,\delta_{jk}\,\delta_{lp} + \delta_{iq}\,\delta_{jl}\,\delta_{kp} + \delta_{iq}\,\delta_{jp}\,\delta_{kl}).\end{aligned}$$

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